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A new class of solutions of the Dorokhov–Mello–Pereyra–Kumar equation

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Abstract

We introduce and discuss a new class of solutions of the Dorokhov–Mello–Pereyra–Kumar (DMPK) equation in which some of the eigenvalues are grouped into clusters which are conserved in the asymptotic large distance limit (i.e. as the length of the wire increases). We give an explicit expression for the asymptotic expansion of these solutions and suggest some possible applications. In particular, these new solutions could be useful for avoiding the quasi-one-dimensional constraint in the DMPK equation and for studying the crossover between the metallic and insulating phases.

1. Introduction

One of the most interesting tools for describing the electron transport properties of quantum wires is the so called Dorokhov–Mello–Pereyra–Kumar (DMPK) equation [1]. This equation describes the evolution of the joint probability distribution of the transmission eigenvalues $P(\{\lambda_i\})$ as the length of the wire increases and has been the subject of intense study in the last few years [2–11]. Among the several remarkable features of this equation, the most interesting one is that it can be exactly mapped onto the radial part of the Laplace–Beltrami operator of suitable symmetric spaces [2]. Thanks to this mapping it is possible to write the Green function of the DMPK equation in terms of the so called zonal spherical functions (ZSFs) of the corresponding symmetric spaces. In the $\beta = 2$ case the ZSF can be written explicitly in terms of ordinary hypergeometric functions [12], thus leading to an exact solution of the DMPK equation [4], while for $\beta = 1, 4$ one can rely on a powerful asymptotic expansion due to Harish-Chandra [13]. This approach leads to a rather involved expression for $P(\{\lambda_i\})$ which however drastically simplifies in the two limits of very short wires (metallic regime) and very long wires (insulating regime), thus allowing us to evaluate all the quantities of interest (see [4] and [5] for a detailed discussion).

However, despite these remarkable results there are two major drawbacks in the DMPK approach to quantum wires. The first is that the DMPK description only holds in the (quasi-)

one-dimensional limit. The second is that the above mentioned solution (even in the simpler $\beta = 2$ case) does not allow us to study the intermediate region between the metallic and the insulating regimes (the so called 'cross-over regime') where no simplifying approximation is allowed. This region has recently attracted much interest since both numerical and analytical results seem to indicate a rather non-trivial (with possibly a non-analytic point [6]) behaviour of $P(\{\lambda_i\})$ exactly in this regime [7–9].

In the last few years several interesting approaches have been proposed to overcome these problems. Various generalizations of the DMPK equation have been suggested [11, 10] to avoid the quasi-one-dimensional limit. However, in all these generalized equations most of the nice properties of the DMPK equation are lost (mainly due to the fact that the description in terms of symmetric spaces is lost) and only a little information on the expected joint probability density of the transmission eigenvalues can be obtained. The aim of this paper is to propose a completely different approach to these problems, which instead fully exploits the power of the symmetric space structure which is behind the DMPK equation. Instead of modifying the DMPK equation, we shall keep it unchanged, but shall look for a set of special solutions (with non-trivial initial conditions) which break the isotropy ansatz. To this end we shall use the fact that the DMPK equation can be mapped into the evolution operator of a class of 1D quantum integrable models known as Calogero–Sutherland (CS) models (for a review see [14]). As a consequence of their exact integrability it is possible to show that in these models, besides the well known symmetric solution, a wide class of non-trivial (but exact) solutions exist in which the particles are grouped into clusters which survive in the asymptotic limit. The clusters of particles become, once the mapping with the DMPK equation is performed, clusters of eigenvalues. The exact integrability of the CS model ensures that this asymmetric distribution of eigenvalues survives in the asymptotic limit and the remarkable properties of the underlying symmetric space allow us to write explicitly such an asymptotic expansion.

This paper is organized as follows: in section 2 we shall recall some known results on quantum transport, focusing on the DMPK equation in order to fix notation. In section 3 we shall review the main concepts associated with CS models and the mathematical tools applied in order to find their solutions. In section 4 we shall describe the new class of solutions of the DMPK equation and in section 5 we shall give some hints on how these new solutions could be used in order to address the open problems mentioned in the introduction.

2. Quantum transport in a wire and DMPK equation

A mesoscopic conductor can be modelled as a disordered region located between two ideal leads connected to two electron reservoirs; at very low temperatures, frequencies and voltages the scattering phenomena inside the wire are supposed to be elastic, and the electron wavefunctions are assumed to be phase coherent. Given a quantum wire of length L and width W ($L \gg W$) it is possible to find a finite but large number N of Fermi channels associated with the conduction electrons. Adopting the notation used in [1], we call the N-dimensional vectors associated with the amplitudes of the incoming and outgoing waves on the left of the wire I and O respectively and the corresponding vectors on the right I' and O'.

The scattering phenomena inside the wire can be described by means of the $2N$-dimensional transfer matrix M, which connects the amplitudes of each channel located on the left to the ones on the right of the wire:

$$M \begin{pmatrix} I \\ O \end{pmatrix} = \begin{pmatrix} O' \\ I' \end{pmatrix}. \tag{1}$$

The matrix M can be decomposed in a radial and an angular part according to the polar decomposition [1]:

$$M = \begin{pmatrix} u_1 & 0 \\ 0 & u_3 \end{pmatrix} \begin{pmatrix} \sqrt{1+\Lambda} & \sqrt{\Lambda} \\ \sqrt{\Lambda} & \sqrt{1+\Lambda} \end{pmatrix} \begin{pmatrix} u_2 & 0 \\ 0 & u_4 \end{pmatrix} \equiv U\Gamma V \tag{2}$$

with u_i $N \times N$ unitary matrices, which assume the role of angular coordinates, and Λ a real diagonal matrix with non-negative elements $\{\lambda_1 \dots \lambda_N\}$, as radial coordinates.

In [1] the general symmetries of the system (time reversal and/or rotational symmetry) fix the matrix M to a particular Lie group:

- if time reversal symmetry (TRS) is present and there is no spin-rotational symmetry (SRS) inside the wire $M \in Sp(2N, \mathbf{R})$;
- if TRS and SRS are both present inside the wire $M \in SO^*(4N)$;
- if TRS is absent $M \in SU(N, N)$.

Experimentally, these three symmetry classes correspond to the presence or absence of a magnetic field (which breaks the TRS) and to the relevance of the spin–orbit interaction term (which breaks SRS) inside the wire.

The general symmetries of the transfer matrix lead to a possible formulation of the problem by means of the tools offered by random matrix theory; each symmetry class is assigned to an ensemble of transfer matrices, associated with a parameter β which can be interpreted as the number of degrees of freedom of each matrix element.

In the framework of the transfer matrix approach to quantum wires the DMPK equation [1] describes the evolution of the probability distribution P_L of the transmission eigenparameters $\{\lambda_i\}$ as a function of the reduced length $s = L/l$ (L being the length of the wire and l the mean free path associated with the electron):

$$\frac{\partial P_L}{\partial s} = \frac{2}{\gamma} \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \lambda_i (1+\lambda_i) J_\beta \frac{\partial}{\partial \lambda_i} J_\beta^{-1} P_L \equiv D P_L \tag{3}$$

with

$$\gamma \equiv \beta N + 2 - \beta$$

where N is the number of Fermi channels associated with the conduction electrons, $\beta = \{1, 2, 4\}$ is the parameter associated with the symmetry of the system, J_β is the Jacobian from the space of the whole matrix M to the space of the eigenparameters $\{\lambda_i\}$,

$$J_\beta(\{\lambda_i\}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} |\lambda_i - \lambda_j|^\beta$$

and D is the DMPK operator. Given the probability distribution P_L, the conductance of the wire can be obtained by means of the Landauer formula [15]. It is important to stress that the construction of the DMPK equation requires an *isotropy ansatz* among the channels. Physically, this implies that the finite time required for the electron to be scattered in the transversal direction is assumed to be infinitesimally small, and this assumption fits only for a quasi-1D wire.

The transfer matrix ensembles which are at the basis of the DMPK equation are very interesting from a mathematical point of view since they can be mapped on a suitable symmetric space of *negative* curvature (see [2] and [3] for a detailed discussion). Similar representations in terms of symmetric spaces also exist for other random matrix theories, but they usually involve symmetric spaces of zero or positive curvature (see [3]). The negative curvature of the space is a peculiar feature of the transfer matrix ensembles and has relevant consequences

Table 1. Symmetry classes for a quantum wire.

β	TRS	SRS	M	X
1	Yes	Yes	$Sp(2N, \mathbf{R})$	$Sp(2N, \mathbf{R})/\mathbf{U}(\mathbf{N})$
2	No	Yes	$SU(N, N)$	$SU(N, N)/SU(N) \otimes SU(N) \otimes U(1)$
4	Yes	No	$SO^*(4N)$	$SO^*(4N)/U(2N)$

on the quantum wire applications of these ensembles. For instance, it can be shown that the eigenparameters $\{\lambda_i\}$ appearing in formula (2) inside Λ exactly correspond to the radial coordinates of the symmetric space while the 'radius' of the space can be mapped (with a suitable normalization) onto the length of the wire. The fact that the radial coordinates in a negative curvature symmetric space flow to infinity as the radius increases then becomes in the quantum wire context the well known result that as the length of the wire increases the wire moves from a conducting to an insulating regime. We shall discuss this mapping in great detail in the next section.

Tables 1 and 2 contain the main symmetry classes associated with the parameter β and the corresponding symmetric space X associated with the eigenparameters $\{\lambda_i\}$.

3. Calogero–Sutherland models: mapping of the DMPK equation on a symmetric space and solutions

In order to address the symmetric space description of quantum wires we need an intermediate ingredient, i.e. the well known CS models (for a review see [14]). These models describe N interacting particles in one dimension, and are characterized by a Hamiltonian that can be mapped, for certain values of the coupling constants, into the radial part of the Laplace–Beltrami operator (B in the following) on a suitable symmetric space X. This operator is the generalization of the familiar Laplace operator describing free diffusion in the Cartesian space and it describes free diffusion on a generic manifold X (in our case X is one of the non-compact symmetric spaces X of tables 1 and 2, characterized by a root lattice structure of type C_N).

The action of the Laplace–Beltrami operator can be formulated in terms of the following eigenvalue equation:

$$B\Phi_k(x) = k^2\Phi_k(x); \tag{4}$$

the eigenfunctions $\Phi_k(x)$ are known in the literature as ZSF (for a review see [3]).

The connection between the CS Hamiltonian and the symmetric space (in particular the root lattice associated with the symmetric space) and the mapping of the CS Hamiltonian into the radial part of the Laplace–Beltrami operator is the key point which determines the integrability of the model.

The general form of the CS Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{\alpha\in\Delta^+} g_\alpha^2 v(x_\alpha) \qquad x = (x_1, \ldots, x_N), \quad p_i = -\mathrm{i}\frac{\partial}{\partial x_i}, \quad x_\alpha = (x, \alpha) \tag{5}$$

where Δ^+ is the set of positive roots α associated with the space X, (x, α) denotes the scalar product, $v(x_\alpha)$ is the interaction potential and g_α is a coupling constant which depends on the roots as

$$g_\alpha^2 = \frac{m_\alpha(m_\alpha - 2)|\alpha^2|}{8} \tag{6}$$

Table 2. Symmetric spaces associated with the transfer matrix and their root multiplicities.

β	X	m_o	m_1	m_s
1	$Sp(2N, \mathbf{R})/\mathbf{U}(\mathbf{N})$	1	1	0
2	$SU(N, N)/SU(N) \otimes SU(N) \otimes U(1)$	2	1	0
4	$SO^*(4N)/U(2N)$	4	1	0

where m_α is the root multiplicity in the space X and $|\alpha^2|$ its length; relation (6) is a necessary constraint for the model to be solvable. The particular CS Hamiltonian we are interested in contains a potential of the form

$$v(x_\alpha) = \frac{1}{\sinh^2 x_\alpha}. \tag{7}$$

The relevant point for the present analysis is that if we choose, as in [4],

$$\lambda_i = \sinh^2 x_i \tag{8}$$

the DMPK operator can be mapped exactly onto a CS Hamiltonian with a potential of the type (7) and the symmetries of one of the symmetric spaces listed in tables 1 and 2.

These spaces are characterized by root lattices (denoted as C_N) composed both by ordinary roots (whose multiplicity $m_o = 1, 2$ or 4 can be identified with the β parameter which encodes the symmetry properties of the matrix model) and long roots with multiplicity $m_1 = 1$ which are responsible for the peculiar properties of the eigenvalues near the boundary $\lambda \sim 0$ (see [2, 3] for further details). Given the set $\{e_j\}$ of linearly independent versors associated with the space, ordinary roots are commonly written as $\alpha = \pm e_j \pm e_i$ $(j \neq i = 1, N)$ and long roots correspond to $\alpha = \pm 2e_j$ $(j = 1, N)$.

These root lattices are characterized by a Z_2 symmetry, i.e. they are invariant under reflection. This means that the origin plays a special role. Indeed, most of the results we shall discuss in the following can be more easily understood by introducing an additional (fictitious) eigenvalue which is kept fixed in the origin and interacts (with a standard repulsive interaction mediated by the long roots of the lattice) with all the other eigenvalues.

The complete series of steps involved in this mapping can be found in [2, 3]. In particular, it turns out that the radial part of the Laplace–Beltrami operator B on the symmetric space X is related to the DMPK evolution operator D in equation (3) by

$$D = \frac{1}{2\gamma}\xi^2(x) B \xi^{-2}(x) \tag{9}$$

where the function $\xi(x)$ is given by

$$\xi(x) = \prod_{i<j} |\sinh^2 x_j - \sinh^2 x_i|^{\beta/2} \prod_i |\sinh 2x_i|^{1/2} \tag{10}$$

and the operator B depends only on the radial coordinates $\{x_i\}$ of the space according to the relation

$$B = [\xi(x)]^{-2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} [\xi(x)]^2 \frac{\partial}{\partial x_i}. \tag{11}$$

As a consequence, if $\Phi_k(x)$, $x = \{x_1, \ldots, x_N\}$, $k = \{k_1, \ldots, k_N\}$ is an eigenfunction of B with eigenvalue k^2, then $\xi(x)^2 \Phi_k(x)$ will be an eigenfunction of the DMPK operator with eigenvalue $k^2/(2\gamma)$.

Once the eigenfunctions of the DMPK operators are known it is rather straightforward to construct the Green function and use it to solve the DMPK equation. The solution turns out to

be rather involved, but it drastically simplifies in the two interesting limits of insulating (small values of k) and metallic (large values of k) regimes. It is important to stress at this point that even if an explicit form is known only for the special case $\beta = 2$, a general and powerful asymptotic expansion for large values of x exists for all the possible symmetric spaces [2, 3]. This asymptotic expansion turns out to be enough for the construction of the two (insulating and conducting) limiting solutions. Its explicit form is

$$\Phi_k(x) \sim \frac{1}{\xi(x)}\left(\sum_{r\in W} c(rk)\mathrm{e}^{\mathrm{i}(rk,x)}\right), \tag{12}$$

where rk is the vector obtained acting with $r \in W$ on k, W is the Weyl group associated with the root system of the symmetric space and (k, x) denotes the scalar product.

All the information related to the underlying symmetric space is encoded in the function $c(k)$ which is given by

$$c(k) = \prod_{\alpha\in\Delta^+} c_\alpha(k) \tag{13}$$

with

$$c_\alpha(k) = \frac{\Gamma(\mathrm{i}(k,\alpha)/2)}{\Gamma(m_\alpha/2 + \mathrm{i}(k,\alpha)/2)} \tag{14}$$

where Γ denotes the Euler gamma function, α is a generic root belonging to the root lattice which defines the symmetric space, m_α denotes its multiplicity and the product is restricted to the sublattice Δ^+ of positive roots only.

4. 'Clustered' solutions of the DMPK equation

In the previous section the integrability of the DMPK equation was stressed in order to write explicitly, at least in the asymptotics, its solutions. In several physical applications (see next section) it would be interesting to study solutions of the DMPK equation in which the symmetry among the N eigenvalues is broken. The aim of this paper is to show how to exploit again the integrable nature of the DMPK equation in order to go beyond the isotropic solutions obtained in [2]. The key point is that, given the integrability of the DMPK equation, its solution can be written even if non-trivial initial conditions are assumed; this observation allows us to construct non-isotropic solutions starting from an equation which is isotropic by construction. The simplest way to obtain this is to impose that some of the eigenvalues form a 'cluster' (i.e. the distances among them remain finite while the distances with respect to the other ones go to infinity), or more generally a set of independent clusters; the peculiar form of the DMPK equation ensures that such solutions exist and survive in the asymptotic limit. These solutions obviously require suitably chosen initial conditions, which are degenerate in the whole phase space. The main goal of this paper is to show that these clustered solutions also admit an asymptotic expansion similar to that of equation (12). Given this expansion, one can then obtain the probability density $P(\{x_n\}, s)$ in the presence of these clusters in a way analogous to the isotropic case. We shall consider below some possible applications of this result.

Let us see these solutions in more detail. Let us assume the cluster to be composed by the first $N' < N$ eigenvalues. This means

$$\begin{aligned} &|x_i - x_j| < \infty, \qquad i, j = 1, \ldots, N' \quad (i \neq j). \\ &|x_i - x_j| \to \infty, \qquad i = 1, \ldots, N; \quad j = N'+1, N \quad (i \neq j). \end{aligned} \tag{15}$$

In the symmetric space framework we can identify the cluster by selecting a subset of the root system associated with the space. Let Π be the system of simple roots associated with X, and Π' be a subsystem of simple roots which satisfies the inequality

$$\Pi' = \left\{ \alpha \in \Pi / \lim_{|x| \to \infty} x_\alpha < \infty \right\} \tag{16}$$

where $x_\alpha = (x, \alpha)$.

There are at this point two possibilities. Since Π is a C_N type lattice then Π' can be again of type C_N (in this case it must also contain the long root and the ordinary roots must be chosen so as to keep the Z_2 symmetry of the lattice) or it can be of type A_N. In both cases from the ordinary roots of Π' one can construct the differences $x_\alpha = x_i - x_{i+1}$ which correspond to the nearest neighbour distances between some of the eigenvalues. From the definition of Π' these distances must remain finite in the asymptotic limit so that Π' defines a cluster (if it is connected) or a set of clusters otherwise. If the cluster is of type A_N there is no other constraint and the cluster can in principle flow to an infinite distance from the origin (while keeping a finite distance among the eigenvalues inside the cluster). If the cluster is of type C_N in contrast, the eigenvalues are bounded (by the lattice structure of Π' itself) to stay within a finite distance from the origin. In the following we shall denote by $\tilde{x}$ the set of radial coordinates outside the cluster and by x' the ones inside the cluster.

The asymptotic expansion of the ZSFs in the presence of such a cluster was obtained a few years ago by Olshanetsky in [16]. It turns out to be a rather natural generalization of the Harish-Chandra result of equation (12):

$$\Psi_k(x) \sim \sum_{r \in W/W'} c_z(\tilde{rk}) \mathrm{e}^{\mathrm{i}(\tilde{rk}, \tilde{x})} \Psi_{(rk)'}(x') \tag{17}$$

where $\Psi_k(x)$ is given by the product $\xi(x)\Phi_k(x)$. In this formula $(rk)'$ denotes the projection of the vector rk on the sublattice Π' and $\tilde{rk}$ its complement, $\xi(x)$ is given by equation (10) and

$$c_z(k) = \prod_{\alpha \in \Delta^+/\Delta'^+} c_\alpha(k) \tag{18}$$

where Δ'^+ is the set of positive roots associated with the cluster, and the W' which appears in the coset W/W' is the Weyl group associated with the cluster. This expression is apparently simple but it is highly non-trivial. Notice for instance that the symmetrization with respect to the Weyl coset W/W' acts not only on the part containing the coordinates $\tilde{x}$ but also on the momenta of the ZSF describing the cluster coordinates $\Phi_{(rk)'}(x')$. This means that the particles inside the cluster do not move independently in a section of the whole space but they 'feel' the presence of the other particles and are subject to the symmetry group of the remaining space. It is interesting to notice that the above construction could also be formulated in the framework of the original transfer matrix ensemble (i.e. before diagonalizing the transfer matrix). The clustered solutions correspond in this framework to peculiar *boundary* limits of the original symmetric spaces (which are known in the mathematical literature as Martin boundaries [17, 18]). The precise characterization of these boundaries is outside the scope of the present paper. We plan to address this issue in a future publication.

Let us see two examples which may hopefully clarify the issue.

Example 1. A C_N type cluster composed by the first $(N-1)$ eigenvalues plus an isolated eigenvalue which flows to infinity, i.e. (see equation (16))

$$\begin{aligned} &|x_i - x_j| < \infty, \qquad i, j = 1, \ldots, N-1 \quad (i \neq j) \\ &|x_i - x_N| \to \infty, \qquad i = 1, \ldots, N-1. \end{aligned} \tag{19}$$

Then the asymptotic form of the ZSF, and of the related product function $\Psi_k(x)$, is

$$\Psi_k(x) \sim \sum_{j=1}^{N} c_z(k_j)\Psi_{\hat{k_j}}(x')\mathrm{e}^{\mathrm{i}k_j x_N} \tag{20}$$

where $\Psi_{\hat{k_j}}(x')$ is associated with the ZSF of the C_{N-1} symmetric space of the cluster, $\hat{k_j} \equiv (k_1, \cdots k_{j-1}, k_{j+1}, \cdots k_N)$ denotes the collection of $(N-1)$ momenta in which k_j is omitted and $c_z(k_j)$ is given in this case by the product of the $c_\alpha(k)$ functions over all the (positive) roots of type $\alpha = \pm e_j \pm e_l$ $(\forall l \neq j)$, $\alpha = 2e_j$ (j fixed). From this expression it is straightforward to construct iteratively the ZSF in which two or more eigenvalues flow to infinity.

Example 2. A C_N type cluster composed by the first two eigenvalues, x_1 and x_2, while the other $(N-2)$ flow to infinity.

$$\Psi_k(x) \sim \sum_{s\in\hat{W}} c_z(k)\Psi_{s(1),s(2)}(x_1, x_2)\mathrm{e}^{\mathrm{i}\sum_{r=3}^{N} k_{s(r)}x_r} \tag{21}$$

where $\hat{W} \equiv W/W'$ is the coset of Weyl groups associated with the whole set of eigenvalues and with the cluster respectively, the subscript of the two-particle wavefunction $\Psi_{s(1),s(2)}(x_1, x_2)$ identifies the two momenta with which this function is associated and $c_z(k)$ in this case is the product of the $c_\alpha(k)$ over all the positive roots of type $2e_{s(m)}$, $2e_{s(n)}$ (with $m, n > 2$) and $\pm e_{s(m)} \pm e_{s(n)}$ (with the only exclusion of the combination $(m, n) = (1, 2)$ or $(2, 1)$).

5. Applications

Among the various possible uses of these new solutions we see in particular four interesting applications.

(1) We can use them to model systems in which the number of open channels is reduced by the structure of the wire itself (see the wide–narrow–wide geometry of [19])[3]. In this case, one could consider a configuration formed by a C_N type cluster (bounded to the origin) made of N' eigenvalues and let the remaining $N - N'$ eigenvalues flow to infinity (see example 1 above). One can choose N' and s (the reduced length of the wire) so as to keep the cluster in the metallic regime, while the other eigenvalues are in the insulating one and do not contribute to the wire conductance.

(2) The same configuration as discussed above could be used to provide a simple but effective way to improve our description of real quantum wires. While in the previous example N' was fixed (being the number of the open channels in the narrow part of the wire) one could easily generalize the example keeping N' as an additional degree of freedom which can be varied so as to take into account the effect of external parameters like the amount of disorder in the wire. The rationale behind this proposal is the well known idea [20] that in a generic conductor only a fraction (which depends on the disorder) of its channels is open and the isotropy anzatz should be a very good approximation for them, while the non-trivial (i.e. non-one-dimensional) behaviour of the conductance is due to the variation in the number of open channels.

(3) The previous picture could be refined choosing configurations with more than one cluster, each cluster being characterized by a different distance at which the crossover between metallic and insulating regimes occurs (fixed by the only free parameter which we have in

[3] We thank C W Beenakker for suggesting this possibility to us.

these generalized solutions, that is the number of eigenvalues of each cluster). This multi-cluster structure could be used in order to evade the isotropy ansatz (this was indeed the main reason for the present investigation). In order to obtain the multi-cluster configuration it is sufficient to choose a *not connected* subset of roots Π' which satisfies the condition given in equation (16). The roots belonging to Π' correspond to a set of coordinates x' which are grouped into different clusters; the number of clusters N_c and their width N_i' are uniquely determined by the choice of the not connected root lattice Π'. The asymptotic expansion given in the general form of equation (17) can then be applied in order to find the proper solutions of this multi-cluster configuration.
The isotropic symmetric space characterized by N radial coordinates in this case is mapped into an anisotropic model in which the N_c channels correspond to N_c clusters (each cluster containing a different number N_i' of coordinates); this reconfiguration should ensure a different probability for the electron to be scattered among the different channels. It is important to stress once again that in our proposal this goal is reached by properly choosing the initial conditions, while the DMPK equation (with all its remarkable properties) *is kept unchanged*. This means, as a side remark, that the DMPK equation still depends on only one parameter i.e. the ratio N/s. The N_c parameters which appear in the special solutions in which we are interested (see the general solution and the examples discussed in section 4) come from the initial conditions and simply consist of the list of the sizes N_i' of the N_c clusters. In this type of solution the N_c channels will encounter the metal–insulator transition at different lengths of the wire; this feature suggests that the N_i'/s parameters can be related to the different mean free paths associated with the channels. This picture recalls the 'non-equivalent channels model' proposed by Mello and Tomsovic in [10] but differs from it since our proposal (as mentioned above) does not require us to modify the DMPK equation.

(4) An interesting independent application is related to the recent attempt [8, 9] to describe the crossover between the metallic and insulating regime in quasi-1D systems by separating out the first [8] or the first two [9] eigenvalues and considering the rest as a continuum in the solution of the DMPK equation. This approximation recalls the clustered solutions that we have discussed in this paper (see in particular example 2). The fact that these clustered configurations are indeed *exact* solutions of the DMPK equation (albeit with non-trivial initial conditions) may explain the remarkable stability of the saddle point solution noticed in [9] and could offer an independent justification for the approach.

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